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On Boltzmann Equations and Fokker–Planck Asymptotics: Influence of Grazing Collisions

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In this paper, we are interested in the influence of grazing collisions, with deflection angle near $\pi/2$, in the space-homogeneous Boltzmann equation. We consider collision kernels given by inverse-sth-power force laws, and we deal with general initial data with bounded mass, energy, and entropy. First, once a suitable weak formulation is defined, we prove the existence of solutions of the spatially homogeneous Boltzmann equation, without angular cutoff assumption on the collision kernel, for $s \geqslant 7/3$. Next, the convergence of these solutions to solutions of the Landau–Fokker–Planck equation is studied when the collision kernel concentrates around the value $\pi/2$. For very soft interactions, $2 < s < 7/3$, the existence of weak solutions is discussed concerning the Boltzmann equation perturbed by a diffusion term.

KEY WORDS: Kinetic equation; homogeneous Boltzmann equation; Landau–Fokker–Planck equation; grazing collisions.

1. INTRODUCTION

This paper is concerned with the spatially homogeneous Boltzmann equation, (hBE),

$$\begin{cases} \partial_t f = Q(f, f) & \text{in } \mathbb{R}_t^+ \times \mathbb{R}_v^3, \\ f|_{t=0} = f_0 & \text{in } \mathbb{R}_v^3 \end{cases} \tag{1.1}$$

which arises in the kinetic theory of gases, assuming that the unknown f, which physically represents a density, does not depend on the space variable. This equation is intended to model the evolution of a rarefied gas

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driven by a binary collision dynamic. The "collision term" is defined by the following quadratic operator

$$Q(f, f) = \int_{\mathbb{R}^3_{v_*} \times S^2_\omega} B(v - v_*, \omega)(f'_* f' - f f_*)\, dv_*\, d\omega \tag{1.2}$$

with the usual notations $f = f(t, v)$, $f_* = f(t, v_*)$, $f' = f(t, v')$, and $f'_* = f(t, v'_*)$ where

$$v' = v - (v - v_*, \omega)\, \omega; \qquad v'_* = v_* + (v - v_*, \omega)\, \omega \tag{1.3}$$

are the post-collisional velocities of two molecules colliding with velocities v and v_*. Let us introduce the following parametrization in the orthonormal basis $((v - v_*)/|v - v_*|, e_2, e_3)$ of $\mathbb{R}^3$

$$\begin{cases} \omega = (\cos(\theta), \sin(\theta)\cos(\psi), \sin(\theta)\sin(\psi)), \\ \cos(\theta) = \dfrac{|(v - v_*, \omega)|}{|v - v_*|}, \theta \in \left[0, \dfrac{\pi}{2}\right], \psi \in [0, 2\pi], \\ d\omega = \sin(\theta)\, d\theta\, d\psi \end{cases} \tag{1.4}$$

The kernel B is a non negative function which only depends on $|v - v_*|$ and on the deflection angle θ. The precise form of B is closely related to the intermolecular potential. We focus our interest on the physically relevant cases where the forces between two particles separated by a distance r take the form $F(r) = 1/r^s$, with $s > 1$. This assumption leads to the following expression for the collision kernel

$$\begin{cases} B(v - v_*, \omega) = |v - v_*|^\gamma\, b(\theta)\, |\cos(\theta)|^{-\nu}, \\ \gamma = \dfrac{s - 5}{s - 1}; \quad \nu = \dfrac{s + 1}{s - 1} \end{cases} \tag{1.5}$$

with, say, $b \in L^\infty([0, \pi/2])$, see refs. 6, 23,.... We recall that, following the terminology introduced in ref. 18, a potential is said "soft" when $s < 5$ and "hard" for $s > 5$; if $s = 5$, we are dealing with a gas of Maxwell's molecules. The case $s = 2$ corresponds to a Coulombian potential which is of particular interest in view of application to plasma physics.[23, 4, 22, 10]

Without additional assumption on b, the kernel (1.5) presents a strong singularity for the so-called "grazing collisions" with angle θ near $\pi/2$. For such collisions the deflection is small and there is pratically no difference between the post-collisional velocities v', v'_* and the velocities v, v_* before the collision. Then, some compensation effects are expected between large values of the impact parameter and the quantity $f'f'_* - ff_*$ which vanishes when $\theta = \pi/2$, so that the integral (1.2) makes sense. Usually, difficulties are avoided by assuming more regularity on the kernel, requiring in particular the integrability of B on the sphere S^2_ω. On a physical viewpoint, such a property is guaranteed when grazing collisions are neglected.[18] This so-called "cutoff assumption" allows us to split the collision term as follows

$$\begin{cases} Q(f, f) = Q^+(f, f) - Q^-(f, f), \\ Q^-(f, f) = fLf; \quad Lf = \int_{S^2} B(\cdot, \omega)\, d\omega *_v f \end{cases} \tag{1.6}$$

The splitting (1.6) remains essential in most of the existence results concerning the Boltzmann equation (see for instance the theory of "renormalized solutions"[13, 24] and the surveys[6, 7]) while very few papers have been published on the Boltzmann equation without the cutoff assumption. Nevertheless, consideration of grazing collisions is performed, for the homogeneous and non homogeneous Boltzmann equations, in ref. 31 where a local in time existence result is established in Gevrey classes, including kernels (1.5) with $s > 3$ and without angular cutoff assumption. In the more general context of integrable solutions, the global existence for (hBE), still with $s > 3$, is discussed in ref. 2.

A related question arises when we are interested in the effect of grazing collisions. Some relations are expected from phenomenological arguments[23, 6] with the Landau–Fokker–Planck equation, (LFP),

$$\begin{cases} \partial_t g = F(g) & \text{in } \mathbb{R}_t^+ \times \mathbb{R}_v^3, \\ g_{|t=0} = f_0 & \text{in } \mathbb{R}_v^3 \end{cases} \tag{1.7}$$

where F is the (nonlinear) differential operator of order 2 defined by

$$F(g) = \nabla \cdot \int |v - v_*|^{\gamma+2} S(v - v_*) \{ g(v_*) \nabla g(v) - g(v) \nabla g(v_*) \}\, dv\, dv_* \tag{1.8}$$

with $S(z) = I - (z \otimes z)/|z|^2$. The formal convergence, for smooth functions, of the operators Q_ε to the operator F, involving an unphysical small parameter ε and functions b_ε which tend to a Dirac mass $\delta_{\theta = \pi/2}$, has been studied in ref. 11, for inverse power force laws with $s > 2$. The Coulombian case, $s = 2$, is considered in ref. 10 where b_ε is a cutoff function and a physical interpretation of ε is given in connection to the "plasma parameter." A first approach of the convergence of solutions of (hBE) to solutions of (LFP) has been outlined in ref. 3, under rather restrictive assumptions on collision kernels and initial data. We also mention that some recent progresses in relation to this subject have been obtained independently in ref. 32 and a part of the results discussed in this paper was announced in refs. 16, 17.

In this paper, we complete and generalize previous works of ref. 2 and ref. 3. First, we introduce a new weak formulation for (hBE), slightly different from those of ref. 2, which allows us to obtain an existence result for inverse power force laws up to $s \geqslant 7/3$, without cutoff assumption on the collision kernel. Next, we consider kernels which concentrate around the value $\theta = \pi/2$, setting in (1.5)

$$b(\theta) = b_\varepsilon(\theta) = \bar{b}\chi_{(\pi/2-\varepsilon,\, \pi/2)}(\theta) \tag{1.9}$$

i.e. almost each collision is grazing. After a suitable time scaling, we study the convergence to (LFP). Our main results are the following.

Theorem 1. Let $f_0 \geqslant 0$ satisfy $(1 + v^2 + |\ln(f_0)|)\, f_0 \in L^1(\mathbb{R}^3)$. We assume that B is given by (1.5) with $7/3 \leqslant s < 5$. If $s \geqslant 5$, we assume moreover $(1+v^2)^r f_0 \in L^1(\mathbb{R}^3)$ with $r > (3s-7)/(2s-2)$. Then, there exists a weak solution f, in the sense of Definition 1, of (1.1) satisfying

$$\sup_{t \geqslant 0} \int (1 + v^2 + |\ln(f)|)\, f\, dv \leqslant C_{f_0} \tag{1.10}$$

where C_{f_0} only depends on f_0.

Theorem 1 justifies the existence of a sequence f_ε of weak solutions of (hBE), associated to kernels B_ε given by (1.5, 1.9). We set, for $s \neq 2$

$$g_\varepsilon(t, v) = f_\varepsilon(F_\varepsilon t, v); \qquad F_\varepsilon^{-1} = \frac{1}{\left|2 - \dfrac{2}{s-1}\right|}\, \varepsilon^{2 - 2/(s-1)} \tag{1.11}$$

Then, (LFP) is derived from (hBE) by letting $\varepsilon \to 0$.

Theorem 2. Let $7/3 \leqslant s \leqslant 3$ and $f_0 \geqslant 0$ satisfy $(1 + v^2 + |\ln(f_0)|)\, f_0 \in L^1(\mathbb{R}^3)$. If $s > 3$, we assume moreover $(1 + v^2)^r f_0 \in L^1(\mathbb{R}^3)$ with $2 \geqslant r \geqslant 2 - 2/(s-1)$. Then, there exist a function $g\colon \mathbb{R}_t^+ \to L_+^1(\mathbb{R}_v^3)$ and a subsequence, still labelled g_ε, such that for a.e. $t > 0$, $g_\varepsilon(t)$ converges weakly in $L^1(\mathbb{R}^3)$ to $g(t)$ and g is a weak solution of (1.7)–(1.8), in the sense of Definition 2.

The results discussed in this work are obtained for general, and physically natural, initial data of bounded mass, energy and entropy; we require a bound on higher moments not exceeding 4 for potentials $s > 3$. (It seems, however, that conditions on higher moments could be relaxed if one considers hard potentials, see refs. 12, 33). The scaling (1.11) involved here indicates that grazing collisions produce non negligible effects on long time and (LFP) retains these contributions. Finally, for very soft interactions, up to the Coulombian potential, an existence result is proved adding a diffusive perturbation term in the Boltzmann equation.

Theorem 3. Let $f_0 \geqslant 0$ satisfy $(1+v^2+|\ln(f_0)|)\, f_0 \in L^1(\mathbb{R}^3)$. We assume that B is given by (1.5) with $2<s<7/3$. Then, there exists a weak solution $f \in C^0(\mathbb{R}_t^+;\, L^1(\mathbb{R}_v^3)) \cap L^2(0, T;\, L^p(\mathbb{R}_v^3))$ with $1 \leqslant p \leqslant 3/2$, in the sense of Definition 3, of the Fokker–Planck–Boltzmann equation (4.1).

This work is organized as follows. Section 2 is devoted to the homogeneous Boltzmann equation without cutoff assumption. In Section 3 we discuss the influence of grazing collisions and the connection to the Landau–Fokker–Planck equation. In Section 4 we complete our existence results for very soft interactions, considering the Fokker–Planck–Boltzmann equation.

2. HOMOGENEOUS BOLTZMANN EQUATION WITHOUT CUTOFF ASSUMPTION

This section is devoted to (hBE), when the collision kernel B is given by (1.5) without cutoff assumption: the function $b\colon [0, \pi/2] \to \mathbb{R}^+$ which appears in (1.5) does not truncate "grazing collisions," we only require that

$$0 \leqslant b(\theta) \leqslant \bar{b}. \tag{2.1}$$

First, we introduce a definition of weak solution, suitabily adapted to (1.1). Next, according to ref. 1, a sequence of approximated problems is constructed. Finally, the existence of weak solution is justified by passing to the limit in the approximated formulations.

We define a weak formulation for the problem (1.1) when the collision kernel presents a singularity given by (1.5). This formulation makes sense requiring only a "natural" L^1 regularity on the solutions f and is obtained reasoning by duality-transposition. Let ϕ be a test function; at least formally one has

$$\begin{aligned}
&\int_{t_0}^{t} \int \partial_t f(\tau, v)\, \phi(\tau, v)\, dv\, d\tau \\
&\quad = -\int_{t_0}^{t} \int f(\tau, v)\, \partial_t \phi(\tau, v)\, dv\, d\tau + \left[\int f(\cdot, v)\, \phi(\cdot, v)\, dv \right]_{t_0}^{t} \\
&\quad = \int_{t_0}^{t} \int Q(f, f)\, \phi\, dv\, d\tau = \int_{t_0}^{t} \int f f_*\, W^{\phi}(v, v_*)\, dv_*\, dv\, d\tau
\end{aligned} \tag{2.2}$$

where

$$W^{\phi}(v, v_*) = \int_{S^2_\omega} B(v - v_*, \omega)(\phi(v') - \phi(v))\, d\omega \tag{2.3}$$

Then, we expand ϕ as

$$\phi(v') - \phi(v) = \nabla\phi(v) \cdot (v' - v) + \int_0^1 D^2\phi(v + u(v' - v)) : (v' - v) \otimes (v' - v)(1 - u)\, du \tag{2.4}$$

where $A : B$ denotes the contracted product $\sum_{i,j} A_{ij} B_{ij}$ of two matrices A and B. We split the right hand side of (2.2) as follows

$$\int Q(f, f)\, \phi\, dv = q(f, \phi) = q^1(f, \phi) + q^2(f, \phi) \tag{2.5}$$

where $q^i(f, \phi)$ involves ith derivatives of the test function ϕ. By using (1.3) and the change of variable $v, v_* \to w_*, w$ one remarks that

$$\begin{aligned} q^1(f, \phi) &= -\int f(v)\, f(v_*)\, \nabla\phi(v) \cdot \int B(v - v_*, \omega)(v - v_*, \omega)\, \omega\, d\omega\, dv\, dv_* \\ &= -\tfrac{1}{2} \int f(v) f(v_*)(\nabla\phi(v) - \nabla\phi(v_*)) \\ &\quad \cdot \int B(v - v_*, \omega)(v - v_*, \omega)\, \omega\, d\omega\, dv\, dv_* \end{aligned} \tag{2.6}$$

Moreover, (1.4) yields

$$\int_0^{2\pi} \omega\, d\psi = 2\pi \cos(\theta) \frac{v - v_*}{|v - v_*|} \tag{2.7}$$

Finally, for a collision kernel given by (1.5), we deduce that

$$q^1(f, \phi) = -\pi I_b \int f(v)\, f(v_*)\, |v - v_*|^{\gamma + 1} (\nabla\phi(v) - \nabla\phi(v_*)) \cdot \frac{v - v_*}{|v - v_*|}\, dv\, dv_* \tag{2.8}$$

where

$$I_b = \int_0^{\pi/2} \frac{b(\theta) \sin(\theta)}{|\cos(\theta)|^{\nu - 2}}\, d\theta \tag{2.9}$$

In particular, it is worth pointing out that the integral I_b is finite for all functions b satisfying (2.1) provided $s > 2$.

The second order term reads as follows

$$q^2(f, \phi) = \int f(v)\, f(v_*)\, |v - v_*|^{\gamma+2} \left\{ \int_0^{\pi/2} \int_0^{2\pi} \int_0^1 D^2\phi(v + u(v' - v)) \right.$$

$$\left. \times (1-u)\, du : \omega \otimes \omega\, d\psi\, \frac{b(\theta) \sin(\theta)}{|\cos(\theta)|^{\nu-2}}\, d\theta \right\} dv\, dv_* \tag{2.10}$$

Let us denote by $C^1_{2,\infty}$ the space of test functions $\phi\colon \mathbb{R}^+_t \times \mathbb{R}^3_v \to \mathbb{R}$ continuously differentiable with compact support in $\mathbb{R}^+_t$, twice continuously differentiable in $\mathbb{R}^3_v$ and such that

$$\|\phi\| = \sup_{t,v} \left| \frac{\phi}{1+v^2} \right| + \sup_{t,v} \left| \frac{\partial_t \phi}{1+v^2} \right| + \sup_{t,v} |D^2_v \phi| < \infty.$$

Previous computations suggest to introduce the following weak formulation for (hBE).

Definition 1. We say that $f\colon \mathbb{R}^+_t \to L^1(\mathbb{R}^3)$ is a weak solution of (1.1), for a collision kernel (1.5), if for all $\phi \in C^1_{2,\infty}$, one has

$$-\int_0^\infty \int f\, \partial_t \phi\, dv\, d\tau - \int f_0 \phi(0, v)\, dv$$

$$= -\pi I_b \int_0^\infty \int f f_*\, |v - v_*|^{\gamma+1}\, (\nabla\phi(v) - \nabla\phi(v_*)) \cdot \frac{v - v_*}{|v - v_*|}\, dv\, dv_*\, d\tau$$

$$+ \int_0^\infty \int f f_*\, |v - v_*|^{\gamma+2}$$

$$\times \left\{ \int_0^{\pi/2} \int_0^{2\pi} \int_0^1 D^2\phi(v + u(v' - v))(1-u)\, du: \right.$$

$$\left. \omega \otimes \omega\, d\psi\, \frac{b(\theta)\sin(\theta)}{|\cos(\theta)|^{\nu-2}}\, d\theta \right\} dv\, dv_*\, d\tau \tag{2.11}$$

with I_b defined by (2.9).

Definition 1 is allowable by remarking that expressions (2.8) and (2.10) make sense for $\phi \in C^1_{2,\infty}$ and $(1+v^2)^r f \in L^1(\mathbb{R}^3)$, with $r \geqslant (3s-7)/(2s-2)$. Indeed, we have

Lemma 1. Let $s \geqslant 7/3$. Then, denoting $\sigma = (3s-7)/(2s-2)$, one has, for $i \in \{1, 2\}$,

$$|q^i(f, \phi)| \leqslant C_s I_b \, \|\phi\| \left(\int (1+v^2)^\sigma f \, dv \right)^2 \tag{2.3}$$

Proof. Obviously, $|\nabla\phi(v) - \nabla\phi(v_*)| \leqslant \|D^2\phi\|_{L^\infty} |v - v_*|$ holds for ϕ in $C^1_{2,\infty}$. Thus, one gets

$$|q^1(f, \phi)| \leqslant \pi I_b \, \|D^2\phi\|_{L^\infty} \int f(v) \, f(v_*) \, |v - v_*|^{\gamma+2} \, dv \, dv_*$$

For $s \geqslant 7/3$, we have $\gamma + 2 = (3s-7)/(s-1) \geqslant 0$, then, by using the elementary inequality $|v - v_*|^{\gamma+2} \leqslant C_s(|v|^{\gamma+2} + |v_*|^{\gamma+2})$ we deduce immediately the asserted bound for $q^1(f, \phi)$. Similar considerations apply to $q^2(f, \phi)$. □

Remark 1. For a weak solution f in the sense of Definition 1, we cannot, in general, split the term $q^1(f, \phi)$ given by (2.8) since $\nabla\phi(v) - \nabla\phi(v_*)$ compensates the singularity with respect to the velocity of the collision kernel. Furthermore, this definition allows us to extend the result of ref. 2 up to $s \geqslant 7/3$ and also will reveal the natural connection to (LFP) in Section 3.

Remark 2. For soft and Maxwell's potentials, $7/3 \leqslant s \leqslant 5$, we have $0 \leqslant \sigma \leqslant 1$. Then, (2.12) can be evaluated by using bounds on mass and energy.

By truncation, we define a sequence of approximations of the collision kernel (1.5)

$$B_n(v - v_*, \theta) = b(\theta) \, |\cos(\theta)|^{-\nu} \, \chi_n(\theta) \, |v - v_*|^{\gamma-1} \, \tilde{\chi}_n(|v - v_*|) \tag{2.13}$$

where we denote by

$$\chi_n(\theta) = \begin{cases} 1 & \text{if } \; 0 \leqslant \theta \leqslant \dfrac{\pi}{2} - \dfrac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$
$$\tilde{\chi}_n(|z|) = \begin{cases} |z| & \text{if } \; \dfrac{1}{n} \leqslant |z| \leqslant n \\ n & \text{if } \; |z| > n \\ 0 & \text{otherwise} \end{cases} \tag{2.14}$$

Obviously, for each n, the kernel B_n is bounded, $0 \leqslant B_n(v-v_*, \theta) \leqslant C_n$ where C_n behaves like $n^{\nu+|\gamma|}$. Then, we recall the following existence result, see [1].

Lemma 2 [1]. Let $f_0 \in L^1_+(\mathbb{R}^3)$ and let B_n be defined by (2.13)–(2.14). Then, there exists a unique solution $f_n\colon \mathbb{R}^+_t \to L^1_+(\mathbb{R}^3_v)$ of (1.1). If, moreover, $v^2 f_0 \in L^1_+(\mathbb{R}^3)$ then, for all $t>0$, we have

$$\int (1, v, v^2)\, f_n(t)\, dv = \int (1, v, v^2)\, f_0\, dv \tag{2.15}$$

and, if $f_0 \ln(f_0) \in L^1(\mathbb{R}^3)$, then $t \mapsto \int f_n(t) \ln(f_n(t))\, dv$ is a non increasing function of time.

Throughout the paper, the initial data f_0 is assumed to satisfy at least

$$\int f_0(1+v^2+|\ln(f_0)|)\, dv < \infty \tag{2.16}$$

so that Lemma 2 yields

$$\sup_{n,\,t} \int f_n(1+v^2+|\ln(f_n)|)\, dv < C_{f_0} \tag{2.17}$$

where the constant C_{f_0} only depends on (2.16) (see for instance ref. 13). Furthermore, it can be shown a useful estimate on higher moments of the solutions, considering hard or Maxwell's potential.

Lemma 3 [15]. Let $s \geqslant 5$ and f_0 satisfy (2.16). Moreover, we assume $|v|^k f_0 \in L^1_+(\mathbb{R}^3)$, $k>2$. Then, we have

$$\sup_{n,\,t} \int |v|^k f_n(t)\, dv \leqslant C_{s,\,f_0} \tag{2.18}$$

where $C_{s,\,f_0}$ only depends on s and on the moments of f_0 of order not exceeding k.

Next, let us establish the following remarkable equi-continuity result.

Lemma 4. Let $7/3 \leqslant s \leqslant 5$ and f_0 satisfy (2.16). If $s>5$, we assume moreover $(1+v^2)^r f_0 \in L^1(\mathbb{R}^3)$ with $r \geqslant (3s-7)/(2s-2)$. Let $\phi \in C^1_{2,\,\infty}$. We set $F_n(t) = \int f_n(t, v)\, \phi(t, v)\, dv$. Then, one has

$$|F_n(t_1) - F_n(t_2)| \leqslant C_{I_b,\,f_0} \|\phi\|\, |t_1 - t_2| \tag{2.19}$$

where $C_{I_b,\,f_0}$ depends on s, I_b and f_0.

Proof. Since f_n satisfies the weak form (2.11), we have

$$|F_n(t_1) - F_n(t_2)| = \left| \int_{t_2}^{t_1} \int f_n(\tau, v)\, \partial_t \phi(\tau, v)\, dv\, d\tau + \int_{t_2}^{t_1} (q_n^1(f_n, \phi) + q_n^2(f_n, \phi))\, d\tau \right|$$

with $q_n^1(f_n, \phi)$, $q_n^2(f_n, \phi)$ defined by (2.8) and (2.10) respectively, where $b(\theta)\, |v - v_*|^\gamma$ is replaced by $b(\theta)\, \chi_n(\theta)\, |v - v_*|^{\gamma - 1}\, \tilde{\chi}_n(|v - v_*|)$. Then, Lemma 1 leads to

$$|F_n(t_1) - F_n(t_2)| \leqslant \|\phi\| \left(\int_{t_2}^{t_1} \int (1 + v^2)\, f_n\, dv\, d\tau + C_s I_b \int_{t_2}^{t_1} \left(\int (1 + v^2)^\sigma f_n\, dv \right)^2 d\tau \right)$$

Hence, for $7/3 \leqslant s \leqslant 5$, since $0 \leqslant \sigma \leqslant 1$, it follows that

$$|F_n(t_1) - F_n(t_2)| \leqslant |t_1 - t_2|\, \|\phi\| \left(\int (1 + v^2)\, f_0\, dv + C_s I_b \left(\int (1 + v^2)\, f_0\, dv \right)^2 \right)$$

by (2.15). If the potential is hard, then $\sigma > 1$ and the asserted estimate is obtained by using the additional assumption on the initial data and Lemma 3

$$|F_n(t_1) - F_n(t_2)| \leqslant |t_1 - t_2|\, \|\phi\| \left(\int (1 + v^2)\, f_0\, dv + C_s I_b C_{s, f_0}^2 \right)$$

with C_{s, f_0} depending on s and on the moments of f_0 of order not exceeding $2r$. □

Combining (2.17) with Lemma 4, one deduces the following compactness property.

Lemma 5. Let $7/3 \leqslant s < 5$ and f_0 satisfy (2.16). If $s \geqslant 5$, we assume moreover $|v|^k f_0 \in L^1_+(\mathbb{R}^3)$ with $k > 2\sigma = (3s - 7)/(s - 1)$. Then, there exist a function f: $\mathbb{R}^+ \to L^1_+(\mathbb{R}^3)$ and a subsequence, still denoted f_n, such that, for each time $t \geqslant 0$ and all ϕ satisfying $|\phi(v)|/(1 + v^2)^\sigma \leqslant C$, one has

$$\lim_{n \to \infty} \int f_n(t)\, \phi\, dv = \int f(t)\, \phi\, dv \tag{2.20}$$

Proof. First, by using the diagonal process of Cantor and the Dunford–Pettis criterion, one deduces from (2.17) that we can extract a subsequence, still labelled f_n, such that for each rational $\bar{t} \geqslant 0$, $f_n(\bar{t})$ converges weakly to a function $f(\bar{t})$ in $L^1_+(\mathbb{R}^3)$. In fact, the bounds on f_n, see Lemma 2–3, imply that convergence (2.20) holds for such $\bar{t}$. Now, pick a non negative time $t \in \mathbb{R}\backslash\mathbb{Q}$. We set $f(t) = \lim_{\bar{t} \in \mathbb{Q} \to t} f(\bar{t})$ in $L^1(\mathbb{R}^3)$. By (2.17), we can assume that $f_n(t)$ converges weakly to a function g in $L^1_+(\mathbb{R}^3)$. The proof is completed by showing that $g = f(t)$. To this end, we use Lemma 4. We set $G(\phi)(t) = \lim_{n \to \infty} F_n(t)$ and $F(\phi)(t) = \int f(t, v)\, \phi\, dv$. Then, for $\phi \in C^1_{2,\infty}$, one has

$$|G(\phi)(t) - F(\phi)(t)| \leqslant |G(\phi)(t) - F_n(\phi)(t)| + |F_n(\phi)(t) - F_n(\phi)(\bar{t})| + |F_n(\phi)(\bar{t}) - F(\phi)(\bar{t})| + |F(\phi)(\bar{t}) - F(\phi)(t)|$$

where $\bar{t}$ is a rational approaching t. Fix $\varepsilon > 0$. By (2.19) and the definition of f, we deduce that

$$\sup_n |F_n(\phi)(t) - F_n(\phi)(\bar{t})| + |F(\phi)(\bar{t}) - F(\phi)(t)| \leqslant \varepsilon$$

holds when $\bar{t}$ is sufficiently close to t. Furthermore, choosing n large enough gives

$$|G(\phi)(t) - F_n(\phi)(t)| + |F_n(\phi)(\bar{t}) - F(\phi)(\bar{t})| \leqslant \varepsilon.$$

Hence, $|G(\phi)(t) - F(\phi)(t)| \leqslant 2\varepsilon$. Thus $G(\phi)(t) = F(\phi)(t)$ for regular function ϕ which leads to $f(t) = g$. □

It remains to pass to the limit $n \to \infty$ in the following weak formulation satisfied by f_n

$$-\int_0^\infty \int f_n \partial_t \phi \, dv\, d\tau - \int f_0 \phi(0,v)\, dv = \int_0^\infty (q^1_n(f_n, \phi) + q^2_n(f_n, \phi))\, d\tau \qquad (2.21)$$

where we recall that, for $i \in \{1, 2\}$,

$$q^i_n(f_n, \phi) = \int f_n(v)\, f_n(v_*)\, W^{\phi, i}_n(v, v_*)\, dv\, dv_* \qquad (2.22)$$

with

$$W_n^{\phi,1}(v, v_*) = -\pi I_n(\nabla\phi(v) - \nabla\phi(v_*)) \cdot \frac{v - v_*}{|v - v_*|} |v - v_*|^\gamma \tilde{\chi}_n(|v - v_*|)$$

$$W_n^{\phi,2}(v, v_*) = \int_0^{\pi/2} \left\{ \int_0^{2\pi} \int_0^1 D^2\phi(v + u(v' - v))(1 - u)\, du\, \omega \otimes \omega\, d\psi \right.$$

$$\left. \times \chi_n(\theta) \frac{b(\theta) \sin(\theta)}{|\cos(\theta)|^{\nu - 2}} d\theta \right\} |v - v_*|^{\gamma + 1} \tilde{\chi}_n(|v - v_*|) \qquad (2.23)$$

and $I_n = \int_0^{\pi/2} \chi_n(\theta)((b(\theta) \sin(\theta))/|\cos(\theta)|^{\nu-2})\, d\theta$.

There is no difficulty to pass to the limit in the left hand side of (2.21). Also it is obvious that $W_n^{\phi,1}$ and $W_n^{\phi,2}$ converge almost everywhere to

$$W_b^{\phi,1}(v, v_*) = -\pi I_b(\nabla\phi(v) - \nabla\phi(v_*)) \cdot \frac{v - v_*}{|v - v_*|} |v - v_*|^{\gamma + 1},$$

$$W_b^{\phi,2}(v, v_*) = \int_0^{\pi/2} \left\{ \int_0^{2\pi} \int_0^1 D^2\phi(v + u(v' - v))(1 - u)\, du : \omega \otimes \omega\, d\psi \right.$$

$$\left. \times \frac{b(\theta) \sin(\theta)}{|\cos(\theta)|^{\nu - 2}} d\theta \right\} |v - v_*|^{\gamma + 2} \qquad (2.24)$$

respectively with

$$\sup_n |W_n^{\phi,1}|,\ \sup_n |W_n^{\phi,2}|,\ |W_b^{\phi,1}|,\ |W_b^{\phi,2}| \leqslant C_s\, \|\phi\|\, I_b(1 + v^2)^\sigma (1 + v_*^2)^\sigma \qquad (2.25)$$

Thus, Lemma 5 implies that, for $i \in \{1, 2\}$,

$$F_n^{\phi,i}(t, v) = \int f_n(t, v_*)\, W_n^{\phi,i}(v, v_*)\, dv_* \to F^{\phi,i}(t, v)$$

$$= \int f(t, v_*)\, W_b^{\phi,i}(v, v_*)\, dv_*$$

a.e. and, moreover

$$|F_n^{\phi,i}|,\ |F^{\phi,i}| \leqslant C_{s, f_0}\, \|\phi\|\, I_b(1 + v^2)^\sigma$$

Hence, the right hand side of (2.21) converges to

$$\int_0^{\infty} (q^1(f, \phi) + q^2(f, \phi))\, d\tau$$

Finally, the proof of Theorem 1 is achieved by remarking that estimate (1.10) is a classical consequence of (2.17) and Lemma 5. We also note that, for $s \geqslant 5$, assuming $|v|^k f_0 \in L^1(\mathbb{R}^3)$ with $k \geqslant 2$, Lemma 3 leads to

$$\int |v|^k f\, dv \leqslant C_{s, f_0} \tag{2.26}$$

Remark 3. *Conservation of energy.* It is clear from (2.26) that a sufficient condition, when $s \geqslant 5$, to obtain the conservation of energy is given by $|v|^{2+\kappa} f_0 \in L^1(\mathbb{R}^3)$, with $\kappa > 0$. However, according to ref. 2, we can prove that, for $3 \leqslant s \leqslant 5$, the conservation of energy holds assuming $v^2 f_0 \in L^1(\mathbb{R}^3)$. Indeed, in this case $\phi(v) = v^2$ satisfies

$$\begin{aligned}
&|v - v_*|^{\gamma+1} f(v)\, f(v_*) \nabla\phi(v) \cdot \frac{v - v_*}{|v - v_*|} \\
&\qquad \leqslant C(1 + |v - v_*|)\, f(v)\, f(v_*)|v| \leqslant C(1 + v^2 + v_*^2)\, f(v)\, f(v_*)
\end{aligned}$$

this last term being integrable on $\mathbb{R}^3 \times \mathbb{R}^3$. It follows from (2.12) that $B(v - v_*, \omega)\, f(v)\, f(v_*)(v'^2 - v^2)$ is integrable on $\mathbb{R}^3 \times \mathbb{R}^3 \times S^2$, therefore $\int Q(f, f)\, v^2\, dv = 0$. Let $\phi_R \in C_0^\infty(\mathbb{R}^3)$ approximate v^2. Then, from the weak form, we obtain

$$\int f(t)\, \phi_R\, dv = \int f_0 \phi_R\, dv + \int_0^t \int Q(f, f)\, \phi_R(v)\, dv\, d\tau$$

Letting $R \to \infty$, we get finally $\int f(t)\, v^2\, dv = \int f_0 v^2\, dv$.

We conclude this section by a discussion on the higher moments of the solutions, concerning soft potentials. This result is adapted to our context from ref. 2 and will be needed in next section.

Lemma 6. Let $7/3 \leqslant s \leqslant 5$ and f_0 satisfy (2.16). If, moreover, $(1 + v^2)^r f_0 \in L^1(\mathbb{R}^3)$ for some $1 \leqslant r \leqslant 2$, then the solution f of (1.1) obtained in Theorem 1 satisfies for all $t > 0$,

$$\int (1 + v^2)^r f(t, v)\, dv \leqslant e^{C_0 I_b t} \int (1 + v^2)^r f_0\, dv \tag{2.27}$$

with C_0 depending only on r and (2.16).

Proof. Let us introduce a test function $\phi_R \in C_0^\infty$ defined by

$$\begin{cases} \phi_R(v) = (1+v^2)^r \text{ on } B(0,R), \operatorname{supp}(\phi_R) \subset B(0,2R), 0 \leqslant \phi_R(v) \leqslant (1+v^2)^r, \\ |\nabla_v \phi_R(v)| \leqslant c(1+v^2)^{r-1/2}, |D_v^2 \phi_R(v)| \leqslant \dot{c}(1+v^2)^{r-1} \end{cases} \tag{2.28}$$

Writing

$$\nabla\phi_R(v) - \nabla\phi_R(v_*) = \int_0^1 D^2\phi_R(v+u(v_*-v))(v_*-v)\,du$$

one gets

$$|q_n^1(f_n, \phi_R)|$$
$$\leqslant I_b \pi \int f_n(v)\, f_n(v_*)\, |v-v_*|^{\gamma+2}\, 8^{r-1} c_r (1+v^2)^{r-1}\, (1+v_*^2)^{r-1}\, dv\, dv_*$$

since $r \geqslant 1$, while the collision kernel is evaluated by using

$$|v-v_*|^{\gamma+2} \leqslant (1+|v-v_*|)^2 \leqslant 4((1+v^2)+(1+v_*^2))$$

for $7/3 \leqslant s \leqslant 5$. This implies that

$$|q_n^1(f_n, \phi_R)| \leqslant I_b \pi 8^r c_r \int f_n(v)(1+v^2)^r\, dv \int f_n(v_*)(1+v_*^2)^{r-1}\, dv_*$$

holds. A similar conclusion can be drawn for $q_n^2(f_n, \phi_R)$. With $1 \leqslant r \leqslant 2$, (1.10) gives

$$\int f_n(t,v)\, \phi_R(v)\, dv$$
$$\leqslant \int f_0(v)\, \phi_R(v)\, dv + \int_0^t |q_n^1(f_n, \phi_R)| + |q_n^2(f_n, \phi_R)|\, d\tau$$
$$\leqslant \int f_0(v)(1+v^2)^r\, dv$$
$$+ I_b 3\pi 8^r c_r \int f_0(v)(1+v^2)\, dv \int_0^t \int f_n(\tau, v)(1+v^2)^r\, dv\, d\tau$$

Letting $R \to \infty$, we deduce that

$$\int (1+v^2)^r f_n(t, v)\, dv \leqslant \int (1+v^2)^r f_0(v)\, dv \exp\left(t.I_b 3\pi 8^r c_r \int (1+v^2)\, f_0(v)\, dv\right)$$

holds by applying Gronwall's lemma. Hence, we conclude by using the weak convergence of $f_n(t)$ to $f(t)$. □

3. INFLUENCE OF GRAZING COLLISIONS, LANDAU–FOKKER–PLANCK EQUATION

According to the existence result proved in Section 2, we can consider a sequence of kernels B_ε which concentrates on grazing collisions. Then, in this section, we wish to investigate the behaviour of the associated weak solutions f_ε of (hBE) when ε goes to 0. First, we define a weak formulation for (LFP), rather close to Definition 1. Next, we introduce a suitable time scaling and, then, we establish the convergence stated in Theorem 2.

Definition 2. We say that $g: \mathbb{R}_t^+ \to L_+^1(\mathbb{R}^3)$ is a weak solution of (1.7)–(1.8), if for all $\phi \in C^1_{3,\infty}$, we have

$$\begin{aligned} -\int_0^\infty \int g\partial_t \phi\, dv\, d\tau - \int f_0 \phi(0, v)\, dv \\ = \int_0^\infty \int g g_* \,|v-v_*|^{\gamma+1} \Big\{ -2(\nabla\phi(v) - \nabla\phi(v_*)) \cdot \frac{v-v_*}{|v-v_*|} \\ + |v-v_*|\, D_v^2\phi(v) : S(v-v_*) \Big\}\, dv\, dv_*\, d\tau \end{aligned} \tag{3.1}$$

Formally, this definition remains equivalent to (1.7)–(1.8), since one has for sufficiently regular functions

$$\begin{aligned} &\int F(g, g)\, \phi\, dv \\ &\quad = -\int |v-v_*|^{\gamma+2} S(v-v_*)(g(v_*)\, \nabla g(v) - g(v)\, \nabla g(v_*)) \cdot \nabla\phi(v)\, dv_*\, dv \\ &\quad = \int g g_* \,|v-v_*|^{\gamma+2} S(v-v_*) : D^2\phi(v)\, dv_*\, dv \\ &\qquad + \int g g_* (div_v - div_{v_*})(|v-v_*|^{\gamma+2} S(v-v_*)) \cdot \nabla\phi(v)\, dv_*\, dv \end{aligned} \tag{3.2}$$

and one remarks that

$$div_v(|v-v_*|^{\gamma+2} S(v-v_*)) = -div_{v_*}(|v-v_*|^{\gamma+2} S(v-v_*))$$
$$= -2\,|v-v_*|^{\gamma+1} \frac{v-v_*}{|v-v_*|} \tag{3.3}$$

Hence, repeating (2.6), we obtain (3.1). In particular, the estimates of Section 2 ensure that the right hand side of (3.1) makes sense for $\phi \in C^1_{2,\infty}$ and $(1+v^2)^r g$ integrable on $\mathbb{R}^3$ with $r \geqslant \sigma' = 2-2/(s-1)$. In Definition 2, we denote by $C^1_{3,\infty}$, the space of test functions $\phi \in C^1_{2,\infty}$, three times continuously derivable in v with bounded third derivatives (the regularity C^3 will be needed later to pass to the limit $\varepsilon \to 0$).

We consider a sequence of collisions kernels defined by

$$\begin{cases} B_\varepsilon(v-v_*, \omega) = |v-v_*|^\gamma\, |\cos(\theta)|^{-\nu}\, b_\varepsilon(\theta), \\ b_\varepsilon(\theta) = \dfrac{2}{\pi}\, \chi_{(\pi/2-\varepsilon,\, \pi/2)}(\theta) \end{cases} \tag{3.4}$$

Such a kernel only sees scattering with deflection angle near $\pi/2$. For $s \geqslant 7/3$, Theorem 1 guarantees the existence of a weak solution f_ε: $\mathbb{R}^+_t \to L^1_+(\mathbb{R}^3_v)$ of (1.1) in the sense of Definition 1. Moreover, we can also deduce from the previous section that (1.10) holds uniformly in ε.

One remarks that, for $s > 2$, the behaviour of I_ε is given by

$$\frac{\pi}{2} I_\varepsilon = \frac{1}{3-\nu} \left|\cos\left(\frac{\pi}{2}-\varepsilon\right)\right|^{3-\nu} \sim \frac{\varepsilon^{3-\nu}}{3-\nu} = \frac{\varepsilon^{2-2/(s-1)}}{2-2/(s-1)} \tag{3.5}$$

Then, we set

$$F_\varepsilon^{-1} = \frac{\varepsilon^{2-2/(s-1)}}{2-2/(s-1)} \tag{3.6}$$

and we introduce the following time scaling

$$g_\varepsilon(t, v) = f_\varepsilon(F_\varepsilon t, v) \tag{3.7}$$

In other words, we are studying the influence of grazing collisions with deflection angle $\pi/2 - \varepsilon \leqslant \theta \leqslant \pi/2$ for time scale of order F_ε.

Estimate (1.10) implies that

$$\sup_{\varepsilon>0}\ \sup_{t\geqslant 0} \int (1+v^2+|\ln(g_\varepsilon)|)\, g_\varepsilon\, dv \leqslant C_{f_0} \tag{3.8}$$

holds for an initial data f_0 satisfying (2.16). Moreover, when $7/3 \leqslant s \leqslant 5$, assuming $(1+v^2)^r f_0 \in L^1(\mathbb{R}^3)$, with $2 \geqslant r > 2 - 2/(s-1)$, Lemma 6 gives

$$\int (1+v^2)^r g_\varepsilon(t, v)\, dv \leqslant C_{r, f_0, T} \tag{3.9}$$

for all $t \in [0, T]$, where $C_{r, f_0, T}$ depends only on r, T and f_0 but not on ε. Such an estimate holds globally in time for $s \geqslant 5$, see (2.26).

Finally, by plugging in (2.11) the following test function

$$\phi_\varepsilon(t, v) = \phi(F_\varepsilon^{-1} t, v) \tag{3.10}$$

with $\phi \in C^1_{2, \infty}$, we are led to

$$-\int_0^\infty \int g_\varepsilon(\tau, v)\, \partial_t \phi(\tau, v)\, dv\, d\tau - \int f_0 \phi(0, v)\, dv$$

$$= \int_0^\infty (\tilde{q}_\varepsilon^1(g_\varepsilon, \phi) + \tilde{q}_\varepsilon^2(g_\varepsilon, \phi))\, d\tau \tag{3.11}$$

where we adopt from now on the following notation

$$\tilde{q}_\varepsilon^1(g_\varepsilon, \phi) = -\pi I_\varepsilon F_\varepsilon \int g_\varepsilon(v)\, g_\varepsilon(v_*)\, |v - v_*|^{\gamma+1}$$

$$\times (\nabla\phi(v) - \nabla\phi(v_*)) \cdot \frac{v - v_*}{|v - v_*|}\, dv_*\, dv \tag{3.12}$$

and

$$\tilde{q}_\varepsilon^2(g_\varepsilon, \phi) = \frac{2F_\varepsilon}{\pi} \int g_\varepsilon(v)\, g_\varepsilon(v_*)\, |v - v_*|^{\gamma+2}$$

$$\times \left\{ \int_{\pi/2-\varepsilon}^{\pi/2} \int_0^{2\pi} \int_0^1 D^2\phi(v + u(v' - v))(1-u)\, du: \right.$$

$$\left. \omega \otimes \omega\, d\psi \frac{\sin(\theta)}{|\cos(\theta)|^{\nu-2}}\, d\theta \right\} dv_*\, dv \tag{3.13}$$

The following lemma legitimate our interest in the convergence when ε goes to 0 in (3.11).

Lemma 7. Let $7/3 \leqslant s \leqslant 5$ and f_0 satisfy (2.16). If $s > 5$, we assume moreover $|v|^k f_0 \in L^1(\mathbb{R}^3)$ with $k \geqslant (3s-7)/(s-1) > 2$. Let $\phi \in C^1_{2,\infty}$. We set $G_\varepsilon(t) = \int g_\varepsilon(t)\, \phi\, dv$. Then, one has

$$|G_\varepsilon(t_2) - G_\varepsilon(t_1)| \leqslant C_{f_0} \|\phi\|\, |t_2 - t_1| \tag{3.14}$$

where C_{f_0} depends on f_0 and s.

Indeed, combining Lemma 7 with estimates (3.8)–(3.9), one deduces the following analogue of Lemma 5.

Corollary 1. Let $7/3 \leqslant s < 5$ and f_0 satisfy (2.16). If $s \geqslant 5$, we assume moreover $|v|^k f_0 \in L^1(\mathbb{R}^3)$ with $k > 2\sigma = (3s-7)/(s-1)$. Then, there exist a function $g\colon \mathbb{R}^+ \to L^1_+(\mathbb{R}^3)$ and a subsequence, still denoted g_ε, such that, for each time $t \geqslant 0$ and all ϕ satisfying $|\phi(v)|/(1+v^2)^\sigma \leqslant C$, one has

$$\lim_{\varepsilon \to 0} \int g_\varepsilon(t)\, \phi\, dv = \int g(t)\, \phi\, dv \tag{3.15}$$

The proof follows arguments in Lemma 4–5 and is omitted.

We achieve the proof of Theorem 2 by passing to the limit $\varepsilon \to 0$ in (3.11). In fact, there is no difficulty to pass to the limit in (3.12) and one gets immediately for the first order term

$$\lim_{\varepsilon \to 0} \tilde{q}^1_\varepsilon(g_\varepsilon, \phi) = -2 \int g(v)\, g(v_*)\, |v - v_*|^{\gamma+1} \times (\nabla\phi(v) - \nabla\phi(v_*)) \cdot \frac{v - v_*}{|v - v_*|}\, dv_*\, dv \tag{3.16}$$

Indeed, $\tilde{q}^1_\varepsilon(g_\varepsilon, \phi)$ reads as follows

$$\tilde{q}^1_\varepsilon(g_\varepsilon, \phi) = F_\varepsilon \int g_\varepsilon(v)\, g_\varepsilon(v_*)\, W^{\phi,1}_\varepsilon(v, v_*)\, dv_*\, dv \tag{3.17}$$

with $W^{\phi,1}_\varepsilon$ defined by (2.24), b_ε replacing b. Then, on the one hand, by the definition (3.5)–(3.6), we have

$$\lim_{\varepsilon \to 0} \pi I_\varepsilon F_\varepsilon = 2 \tag{3.18}$$

and, on the other hand,

$$F_\varepsilon\, |W^{\phi,1}_\varepsilon(v, v_*)| \leqslant \|\phi\|\, C_s (1+v^2)^\sigma (1+v_*^2)^\sigma \tag{3.19}$$

This is sufficient to justify (3.16). To deal with $\tilde{q}_\varepsilon^2(g_\varepsilon, \phi)$ given by (3.13), we expand ϕ up to order 3

$$\phi(v') - \phi(v) = \nabla\phi(v) \cdot (v' - v) + \frac{1}{2} D^2\phi(v) : (v' - v) \otimes (v' - v) + \int_0^1 \frac{(1-u)^2}{2} D^3\phi(v + u(v' - v))(v' - v)^{\otimes 3}\, du \quad (3.20)$$

Therefore, we note that

$$\int_0^{2\pi} \omega \otimes \omega \, d\psi = \pi S(v - v_*) + \pi \cos^2(\theta)(2I - 3S(v - v_*)) \quad (3.21)$$

Then, we split

$$\tilde{q}_\varepsilon^2(g_\varepsilon, \phi) = \tilde{q}_\varepsilon^{2,1}(g_\varepsilon, \phi) + \tilde{q}_\varepsilon^{2,2}(g_\varepsilon, \phi) + \tilde{q}_\varepsilon^R(g_\varepsilon, \phi) \quad (3.22)$$

where

$$\tilde{q}_\varepsilon^{2,1}(g_\varepsilon, \phi) = \frac{\pi}{2} F_\varepsilon I_\varepsilon \int g_\varepsilon(v)\, g_\varepsilon(v_*)\, |v - v_*|^{\gamma+2} D^2\phi(v) : S(v - v_*)\, dv_*\, dv$$

$$\tilde{q}_\varepsilon^{2,2}(g_\varepsilon, \phi) = F_\varepsilon J_\varepsilon \int g_\varepsilon(v)\, g_\varepsilon(v_*)\, |v - v_*|^{\gamma+2} \times D^2\phi(v) : (2I - 3S(v - v_*))\, dv_*\, dv \quad (3.23)$$

with

$$J_\varepsilon = \int_{\pi/2-\varepsilon}^{\pi/2} \frac{\sin(\theta)}{|\cos(\theta)|^{\nu-4}}\, d\theta \quad (3.24)$$

and $q_\varepsilon^R(g_\varepsilon, \phi)$ is a remainder term depending on third derivatives of ϕ. The behaviour of $\tilde{q}_\varepsilon^{2,1}(g_\varepsilon, \phi)$ when ε tends to 0 allows us to complete (*LFP*), since

$$\lim_{\varepsilon \to 0} \tilde{q}_\varepsilon^{2,1}(g_\varepsilon, \phi) = \int g(v)\, g(v_*)\, |v - v_*|^{\gamma+2} D^2\phi(v) : S(v - v_*)\, dv_*\, dv \quad (3.25)$$

Indeed, $\tilde{q}_\varepsilon^{2,1}(g_\varepsilon, \phi)$ takes the simple form

$$\tilde{q}_\varepsilon^{2,1}(g_\varepsilon, \phi) = \frac{\pi}{2} F_\varepsilon \int g_\varepsilon(v)\, g_\varepsilon(v_*)\, W_\varepsilon^{\phi,2}(v, v_*)\, dv_*\, dv \quad (3.26)$$

with,

$$F_\varepsilon\,|W_\varepsilon^{\phi,2}(v,v_*)| = F_\varepsilon I_\varepsilon\,|v-v_*|^{\gamma+2}\,|D^2\phi(v):S(v-v_*)|$$
$$\leqslant \|\phi\|\,C_s\,(1+v^2)^\sigma\,(1+v_*^2)^\sigma \tag{3.27}$$

due to (3.18). We conclude by (3.15). It remains to show that $\tilde{q}_\varepsilon^{2,2}(g_\varepsilon,\phi)$ and $\tilde{q}_\varepsilon^R(g_\varepsilon,\phi)$ tend to 0 with ε. Indeed, the following inequality

$$|\tilde{q}_\varepsilon^{2,2}(g_\varepsilon,\phi)| \leqslant C_s F_\varepsilon J_\varepsilon\,\|\phi\|\left(\int (1+v^2)^\sigma g_\varepsilon\,dv\right)^2 \tag{3.28}$$

holds with J_ε given by (3.24) and, furthermore, we get

$$|\tilde{q}_\varepsilon^R(g_\varepsilon,\phi)| \leqslant C_s F_\varepsilon K_\varepsilon\,\|D^3\phi\|_{L^\infty}\left(\int (1+v^2)^{\sigma'} g_\varepsilon\,dv\right)^2 \tag{3.29}$$

where $\sigma' = 2 - 2/(s-1)$ and

$$K_\varepsilon = \int_{\pi/2-\varepsilon}^{\pi/2} \frac{\sin(\theta)}{|\cos(\theta)|^{\nu-3}}\,d\theta \tag{3.30}$$

Since $F_\varepsilon K_\varepsilon$ and $F_\varepsilon J_\varepsilon$ behave as ε and ε^2 respectively for small values of ε, $\tilde{q}_\varepsilon^{2,2}(g_\varepsilon,\phi)$ and $\tilde{q}_\varepsilon^R(g_\varepsilon,\phi)$ tend to 0 provided $(1+v^2)^\sigma g_\varepsilon$ and $(1+v^2)^{\sigma'} g_\varepsilon$ are bounded in $L^1(\mathbb{R}^3)$. For $7/3 \leqslant s \leqslant 3$, we have $0 \leqslant \sigma' \leqslant 1$, and it is sufficient to assume (2.16) which implies (3.8). For $s>3$, we require moreover the integrability of $(1+v^2)^r f_0$ with $2 \geqslant r \geqslant 2-2/(s-1)$ to conclude by using (3.9). Then, the proof of Theorem 2 is complete. □

4. VERY SOFT INTERACTIONS: FOKKER–PLANCK–BOLTZMANN EQUATION

In this section, we are dealing with the Boltzmann equation, perturbed by a diffusive Fokker–Planck term, still in the space homogeneous context, namely

$$\partial_t f = Q(f,f) + \nu\,\Delta f \qquad \text{in } \mathbb{R}_t^+ \times \mathbb{R}_v^3 \tag{4.1}$$

with $\nu > 0$. We consider very soft interactions up to the Coulombian potential, namely $2 < s < 7/3$. Indeed, (2.9) suggests that the angular singularity is controlled for $s>2$, while, for very soft interactions, main difficulties arise from the singularity with respect to the velocity in the collision kernel.

The weak convergence $f_n \rightharpoonup f$ in L^1 is not sufficient to pass to the limit in terms (2.22)–(2.23) which behave as

$$\int f_n(v)\, f_n(v_*)\, |v - v_*|^{\gamma+2}\, dv\, dv_*$$

where $\gamma + 2 < 0$ for $s < 7/3$. The introduction of the diffusion term $\nu \Delta f$ allows us to obtain some bounds in L^p spaces with $p > 1$ large enough to justify the passage to the limit. Moreover, we expect that the Boltzmann operator is close, in some sense, to the Landau–Fokker–Planck operator when taking into account grazing collisions. Then, our approach is also motivated by the fact that a linearization of the *LFP* operator leads to the diffusion term $\nu \Delta f$, see ref. 19, ref. 20 and the references therein. Keeping the notations of Section 2, we define the weak formulation for (4.1) in a natural way from Definition 1.

Definition 3. We say that $f: \mathbb{R}_t^+ \to L^1(\mathbb{R}^3)$ is a weak solution of (4.1), for a collision kernel (1.5), if for all $\phi \in C^1_{2,\infty}$, one has

$$\begin{aligned} -\int_0^\infty \int f \partial_t \phi\, dv\, d\tau - \int f_0 \phi(0, v)\, dv \\ = \int_0^\infty \left(q^1(f, \phi) + q^2(f, \phi) + \nu \int f \Delta\phi\, dv\right) d\tau \end{aligned} \tag{4.2}$$

We shall prove the existence of such a weak solution. First, let us recall some a priori estimates on the solutions of (4.1). These estimates provide some additional compactness on sequences f_n of (approximated) solutions and, then, we will pass to the limit $n \to \infty$.

By integrating (4.1) with respect to the variable v, one obtains the classical properties of mass conservation $d/dt \int f\, dv = 0$, momentum conservation $d/dt \int vf\, dv = 0$ and increase of kinetic energy $d/dt \int v^2 f\, dv = 6\nu \int f_0\, dv$. By using the equality $\nabla(\sqrt{f}) = \nabla f / 2\sqrt{f}$, we also get

$$\frac{d}{dt} \int f \ln(f)\, dv + \int H(f)\, dv + 4\nu \int |\nabla(\sqrt{f})|^2\, dv = 0$$

where $\int H(f)\, dv = \int B(v - v_*, \omega)(f' f'_* - f f_*) \ln(f' f'_* / f f_*)\, d\omega\, dv_*\, dv \geqslant 0$. Then, for an initial data f_0 of finite mass, energy and entropy we are led to the following bounds for f on $0 \leqslant t \leqslant T < \infty$,[14]

$$\begin{cases} \int (1+v^2+|\ln(f)|)\, f\, dv \leqslant C_{f_0,T,\nu}, \\ \int_0^t \int |\nabla(\sqrt{f})|^2\, dv\, ds \leqslant C_{f_0,T,\nu} \end{cases} \tag{4.3}$$

where $C_{f_0,T,\nu}$ denotes a constant depending only on f_0, T, ν. Let us define the following functional space

$$X = \left\{ \phi : \mathbb{R}^3 \to \mathbb{R}, \int (1+v^2)\, |\phi|\, dv < \infty, \int |\nabla\phi|\, dv < \infty \right\}$$

Therefore, following ref. 5, one has

$$X \subset W^{1,1}(\mathbb{R}^3) \subset BV(\mathbb{R}^3) \subset L^p(\mathbb{R}^3)$$

for $1 \leqslant p \leqslant 3/2$. We deduce that

$$f \text{ is bounded in } L^2(0, T; X) \subset L^2(0, T; L^p(\mathbb{R}^3)) \tag{4.4}$$

Indeed, it is known from (4.3) that f is bounded in $L^2(0, T; L^1(\mathbb{R}^3))$. It remains to estimate

$$\int_0^T \left| \int |\nabla(f)|\, dv \right|^2 ds \leqslant 4 \int f_0\, dv \int_0^T \int |\nabla(\sqrt{f})|^2\, dv\, ds \leqslant C_{f_0,T,\nu}$$

by combining the Cauchy–Schwartz inequality and (4.3).

Next, our aim is to establish an estimate on $\partial_t f$ in a suitable space of distributions. To this end, we use the weak formulation (4.2) which yields

$$\langle \partial_t f, \phi \rangle_{\mathscr{D}', \mathscr{D}(\mathbb{R}^3)} = q^1(f, \phi) + q^2(f, \phi) + \nu \int f \Delta\phi\, dv$$

It follows that

$$|\langle \partial_t f, \phi \rangle| \leqslant \|D^2\phi\|_{L^\infty} \left(2\pi I_b \int f f_* \, |v - v_*|^{\gamma+2}\, dv\, dv_* + \nu \int f\, dv \right) \tag{4.5}$$

holds with $-1 < \gamma + 2 < 0$ for $2 < s < 7/3$. However, by using the Hardy–Littlewood inequality,[28] we have

$$\int \frac{f f_*}{|v - v_*|^{-(\gamma+2)}}\, dv\, dv_* \leqslant C_\gamma \, \|f\|^2_{L^p(\mathbb{R}^3)} \tag{4.6}$$

where $\bar{p} > 1$ is given by $2/\bar{p} - (\gamma + 2)/3 = 2$, i.e.

$$\bar{p} = 1 + \frac{7 - 3s}{9s - 13} \tag{4.7}$$

We note that $1 < \bar{p} < 3/2$ for all $9/5 < s < 7/3$. Let us denote by $\mathscr{Y} = \mathscr{E}^2(\mathbb{R}^3)$ the space of distributions T of order 2 on $\mathbb{R}^3$ which satisfy

$$\|T\|_{\mathscr{Y}} = \sup \left\{ \frac{|\langle T, \phi \rangle|}{\|\phi\|_{C_b^2}},\ \phi \in C_0^\infty(\mathbb{R}^3) \right\} < \infty$$

By (4.6), estimate (4.5) becomes

$$\|\partial_t f\|_{\mathscr{Y}} \leqslant 2\pi I_b C_\gamma \|f\|_{L^{\bar{p}}}^2 + \nu \|f\|_{L^1} \tag{4.8}$$

Then, by combining (4.3), (4.4) and (4.8), we are led to

$$\|\partial_t f\|_{L^1(0,T;\mathscr{Y})} \leqslant 2\pi I_b C_\gamma \|f\|^2_{L^2(0,T;L^{\bar{p}}(\mathbb{R}^3))} + \nu \|f\|_{L^1(0,T;L^1(\mathbb{R}^3))} \leqslant C_{f_0,\nu,T}. \tag{4.9}$$

By truncation and regularization of the collision term, one may consider a sequence f_n of approximated solutions to the Fokker–Planck–Boltzmann equation (4.1), see for instance ref. 14 (see also refs. 19, 20). By (4.3), this sequence is weakly compact in $L^r(0, T; L^1(\mathbb{R}^3))$; however, such a compactness property is not sufficient to pass to the limit $n \to \infty$ in the weak formulation (4.2). The diffusion term gives additional properties on the sequence f_n. Indeed, previous computations prove the following

Lemma 8. The sequence f_n is bounded in $L^2(0, T; X)$ and $\partial_t f_n$ is bounded in $L^1(0, T; \mathscr{Y})$.

However, X embeds compactly in $L^q(\mathbb{R}^3) \subset \mathscr{Y}$ for $1 \leqslant q < 3/2$, ref. 5. We apply Corollary 4, p. 85 of ref. 30, to deduce the following essential compactness property.

Corollary 2. The sequence f_n is compact in $L^2(0, T; L^q(\mathbb{R}^3))$ for $1 \leqslant q < 3/2$.

Corollary 2 allows us to consider a sequence $f_n \to f$ in $L^2(0, T; L^q(\mathbb{R}^3))$, for $1 \leqslant q < 3/2$; in particular the convergence holds for $q = \bar{p}$ given by (4.7). To show that the limit f is a solution of (4.1) in the sense of Definition 3, it remains to pass to the limit in the collision terms written as in (2.22)–(2.23),

$$q_n^i(f_n, \phi) = \int f_n(v)\, F_n^{\phi, i}(v)\, dv \tag{4.10}$$

with

$$F_n^{\phi,i}(v) = \int f_n(v_*)\, W_n^{\phi,i}(v, v_*)\, dv_* \tag{4.11}$$

Obviously, one has $W_n^{\phi,i}(v, v_*) \to W^{\phi,i}(v, v_*)$ a.e. $\mathbb{R}^3 \times \mathbb{R}^3$ and

$$|W_n^{\phi,i}(v, v_*)|,\ |W^{\phi,i}(v, v_*)| \leqslant CI_b\, \|D^2\phi\|_{L^\infty}\, |v - v_*|^{\gamma+2} \tag{4.12}$$

Then, one remarks that

$$\begin{aligned} \|F_n^{\phi,i}\|_{L^r(\mathbb{R}^3)} &\leqslant CI_b\, \|D^2\phi\|_{L^\infty} \left\| \int \frac{f_n(v)}{|v - v_*|^{-(\gamma+2)}}\, dv \right\|_{L^r(\mathbb{R}^3)} \\ &\leqslant C_\gamma I_b\, \|D^2\phi\|_{L^\infty}\, \|f_n(v)\|_{L^p(\mathbb{R}^3)} \end{aligned} \tag{4.13}$$

holds for $1/r = 1/p - 1 + (-(\gamma+2)/3)$, see refs. 21, 29. In particular, the choice $r = p'$ (Holder's conjugate of p) leads to $p = \bar{p}$. One deduces that

$$F_n^{\phi,i} \to F^{\phi,i} = \int f(v_*)\, W^{\phi,i}(v, v_*)\, dv_* \tag{4.14}$$

strongly in $L^2(0, T; L^{\bar{p}'}(\mathbb{R}^3))$. This allows us to pass to the limit $n \to \infty$ in (4.10) and gives (4.2) which ends the proof of Theorem 3. □

CONCLUDING REMARKS

A similar analysis can be drawn for the linear Boltzmann equation (and also, certainly, for the linearized Boltzmann equation, according to ref. 11). The simplicity of the collision operator allows us to consider more singular kernels and the results obtained in ref. 17 (where the existence results of refs. 25–27, 8 are extended up to $s > 2$) may indicate what kind of behaviour we can expect for the Boltzmann equation (1.1). Indeed, it is worth pointing out that the study of the influence of grazing collisions reveals different behaviour depending on the position of s with respect to the critical Coulombian case. Namely, for $s > 2$, *concentrations* of the collision kernel on $\pi/2$, produce effects described by the Fokker–Planck equation on a *large* time and space scale. On the contrary, for Coulombian and very soft interactions $9/5 < s \leqslant 2$, *truncations*, neglecting grazing collisions, lead to the Fokker–Planck equation on a *short* scale (logarithmic in the Coulombian case). These results are in agreement with physical analysis which explain that, for Coulombian interaction, collisions with large impact parameter have a dominating influence,[23,10] and require to take into account some screening effects in the modeling of the interactions.[22,4]

We also refer to ref. 10 and ref. 32 for further developments in this direction.

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